# Homogeneous Equations

**Def:** a function $f(x,y)$ is <u>homogeneous of degree $n$</u> if

$$f(tx,ty) = t^n f(x,y) \text{ for any } t \in \mathbb{R}.$$

Ex: $f(x,y) = x^2+xy-y^2$ is homogeneous of degree 2 because

$$f(tx,ty) = (tx)^2+(tx)(ty)-(ty)^2$$
$$= t^2\{x^2+xy-y^2\}$$
$$= t^2 f(x,y)$$

**Def:** a first order ODE,

$$f(x,y)dx + g(x,y)dy = 0$$

is <u>homogeneous</u> if $f(x,y)$ and $g(x,y)$ are both homogeneous of the same degree.

In this case we can write the equation as

$$\frac{dy}{dx} = \frac{-f(x,y)}{g(x,y)}$$

and $\dfrac{dy}{dx} = \dfrac{-f(tx,ty)}{g(tx,ty)} = \dfrac{-t^n f(x,y)}{t^n g(x,y)} = \dfrac{-f(x,y)}{g(x,y)}$

in particular, if we let $t = \frac{1}{x}$ we find

$$\frac{dy}{dx} = \frac{-f(x,y)}{g(x,y)} = -\frac{(\frac{1}{x})^n f(x,y)}{(\frac{1}{x})^n g(x,y)} = -\frac{f(\frac{x}{x},\frac{y}{x})}{g(\frac{x}{x},\frac{y}{x})} = -\frac{f(1,y/x)}{g(1,y/x)}$$

and the derivative can be written as a function of the single variable $y/x$.

This suggests that if we make the substitution

$$v = y/x$$

we can transform $\dfrac{f(x,y)}{g(x,y)}$ to a function of the single variable $v$.

The left side, $\dfrac{dy}{dx}$ can be expressed in terms of $v$ by one application of the product rule –

$$y/x = v$$
$$y = xv$$

and $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$

The resulting equation

$$x\frac{dv}{dx} + v = \frac{-f(1,v)}{g(1,v)} = \frac{-f(v)}{g(v)}$$

is then separable.

Ex: Find the general solution of

$$\frac{dy}{dx} = \frac{y+x}{x}$$

In practice, we only ask if $\frac{y+x}{x}$ can be written in terms of the single variable $y/x$. In this case it can -

$$\frac{y+x}{x} = \frac{y}{x} + 1$$

so we let $v = y/x$, $\frac{dy}{dx} = x\frac{dv}{dx} + v$ and find

$$x\frac{dv}{dx} + v = v + 1$$

This equation is separable -

$$x\frac{dv}{dx} + v = v + 1$$

$$x\frac{dv}{dx} = 1$$

$$dv = \frac{dx}{x}$$

and $v = \ln|x| + c$

of course, we need to express the solution in terms of the original variables $x$ & $y$ so

$$\frac{y}{x} = \ln|x| + c$$

or $\boxed{y = x\ln|x| + cx}$

or $y = x(\ln|x| + c)$

$\boxed{y = x\ln|cx|}$ if we notice that $c = \ln|c|$

ex: Find the general solution of $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$.

Again, we don't even need the definition of a homogeneous function. We only care whether we can write the equation solely in terms of the quotient $y/x$. In this case we clearly can because

$$\frac{x^2+y^2}{xy} = \frac{1+(y/x)^2}{(y/x)}$$

So the equation becomes –

$$\frac{dy}{dx} = \frac{1+(y/x)^2}{(y/x)}$$

We substitute $v = y/x$

and $y = xv$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

The equation is then

$$x\frac{dv}{dx} + v = \frac{1+v^2}{v}$$ which is separable –

$$\frac{x\,dv}{dx} = \frac{1+v^2-v^2}{v}$$

$$v\,dv = \frac{dx}{x}$$

$$\frac{v^2}{2} = \ln|x| + c$$

$$v^2 = 2\ln|x| + C \quad (\text{where } 2c = C)$$

$$v^2 = \ln(x^2) + C$$

We now express in terms of the original variables and find

$$\frac{y^2}{x^2} = \ln(x^2) + C$$

$$\boxed{y^2 = x^2\ln(x^2) + Cx^2}$$ which is the general solution.

ex: Solve the IVP $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$, $y(1) = -2$

We have the general solution $y^2 = x^2\ln(x^2) + Cx^2$. Applying the initial condition $y(1) = -2$ gives

$$4 = \ln(1) + C, \quad C = 4, \text{ and}$$

$$y^2 = x^2\ln(x^2) + 4x^2$$

We can express this as one explicit solution $\boxed{y = -\sqrt{x^2\ln(x^2) + 4x^2}}$

You should show that the other solution $y = \sqrt{x^2\ln(x^2) + 4x^2}$ is not a solution (it does not satisfy $y(1) = -2$).

There is nothing special about the substitution $v = y/x$. In fact, if we can make the substitution $v = y/x$ we can just as well make the substitution $v = x/y$.

In the previous problem we have

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

we can invert and we have

$$\frac{dx}{dy} = \frac{xy}{x^2+y^2}$$

$$= \frac{(x/y)}{(x/y)^2+1} \quad \text{by dividing numerator \& denominator by } y^2.$$

So we can make the substitution $v = \frac{x}{y}$

and $x = yv$

$$\frac{dx}{dy} = y\frac{dv}{dy} + v$$

Then $y\frac{dv}{dy} + v = \frac{v}{v^2+1}$ which is separable.

$$y\frac{dv}{dy} = \frac{-v^3}{v^2+1}$$

$$-\frac{v^2+1}{v^3}\,dv = \frac{dy}{y}$$

integrating, we have

$$-\ln|v| + \frac{1}{2v^2} = \ln|y| + c$$

$$\frac{1}{2v^2} = \ln|yv| + c$$

$$\frac{1}{v^2} = 2\ln|yv| + c \quad (2c = c)$$

$$\frac{1}{v^2} = \ln(y^2v^2) + c$$

Putting this solution in terms of $x$ & $y$ we find

$$\frac{1}{(x/y)^2} = \ln\left(y^2 \cdot \frac{x^2}{y^2}\right) + c$$

$$\frac{y^2}{x^2} = \ln(x^2) + c$$

or $\boxed{y^2 = x^2\ln(x^2) + cx^2}$

Which is the same solution we already found.